

On Semimartingale local time inequalities and Applications in SDE's

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Abstract

Using the balayage formula, we prove an inequality between the measures associated to local times of semimartingales. Our result extends the "comparison theorem of local times" of Ouknine (1988), which is useful in the study of stochastic differential equations. The inequality presented in this paper covers the discontinuous case. Moreover, we study the pathwise uniqueness of some stochastic differential equations involving local time of unknown process.

Keywords: Semimartingales, local times, stochastic differential equation, balayage formula, Skorokhod problem.

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1 Introduction

In contrast of ordinary differential equation, the theory of stochastic differential equations has two distinguished notions of solutions and two uniqueness properties: Pathwise uniqueness (P.U) and uniqueness in law (U.L). Roughly, the pathwise uniqueness asserts that two solutions on the same probability space with the same stochastic inputs agree almost surely while weak uniqueness asserts that two solutions agree in distributions (Precise definitions will be given later).

On his seminal work, S. Nakao [17] has proved a pathwise uniqueness property for SDE with non regular datas. To do this he used a key lemma on martingale and bounded variation processes. (See an elegant proof of the result in N. V. Krylov and A. K. Zvonkin [13]).

In many references studying the existence and uniqueness of strong solutions of the Itô equations, the common idea is the construction of weak solutions together with the subsequent use of the celebrated pathwise uniqueness argument, obtained by Yamada and Watanabe [30] which can be formulated as:

$$\text{weak existence} + \text{pathwise uniqueness} \Rightarrow \text{strong uniqueness}.$$

For this reason, many investigations were devoted to the problem of (P.U) of solutions of SDE's.

In this paper we are interested by the stochastic differential equations of the form:

$$X_t = X_0 + \int_0^t \sigma_s(X_s, B_s) dB_s + \int_{\mathbb{R}} L_t^a(X) \nu(da) \quad (1)$$

where the mapping function $\sigma : \mathbb{R}^+ \times C(\mathbb{R}^+, \mathbb{R}) \times C(\mathbb{R}^+, \mathbb{R}) \rightarrow \mathbb{R}$ is measurable and adapted to the filtration $(\mathcal{C}_t)_{t \geq 0}$, $L_t^a(X)$ stands for local time of a continuous semimartingale X at a , ν is σ -finite measure on \mathbb{R} . This equations was studied by H. J. Engelbert [8], the particular case where $(\sigma_t)_{t \geq 0}$ is given by a borel function σ defining

$$\sigma_t(x, w) = \sigma(x_t) \quad \text{for all } (x, w) \in C(\mathbb{R}^+, \mathbb{R}^2)$$

was considered by several authors, c.f., for example, Le Gall [15], H. J. Engelbert and W. Schmidt [9] and recently by R. Belfadli and Y. Ouknine [4]. Using the local time technique and the inequality between local times of continuous semimartingales given by Y. Ouknine [20], we prove the path-wise uniqueness for the SDE (1). In fact, this equality which was extended to discontinuous case by F. Coquet and Y. Ouknine [7] is very useful in the study of SDE, several works was done in litterature, e.g. Y. Ouknine [20], Belfadli and Ouknine [4], Rutkowski [29]. So, in the first part of this paper, we prove a general result for comparison theorem for local times. Roughly speaking, if X and Y are two semimartingales sharing the same set of zero, and if $X^+ \leq Y^+$, then the measure $dL^0(X)$ is absolutely continuous with respect to the measure $dL^0(Y)$.

Our proof is based on the use of balayage formula of Azéma and Yor [1], which is the key of the first order calculus. And which proves an efficient tool to obtain the comparison theorem for local times in its strong form.

The structure of this paper is as follows:

Section(2): We start it with the continuous version of the balayage formula and show how to deduce from it our new comparison theorem for local times of continuous semimartingales.

Section (3): Illustrate our main result for the càdlàg semimartingales.

Section (4) : It contains three subsections concentrating to on the study of the pathwise uniqueness of some classes of SDE's.

Subsection (4.1): We give necessary and sufficient condition for pathwise uniqueness of solutions for SDE's with local time component:

$$dY_t = \sigma(Y_t) dB_t + b(Y_t) dt + \frac{1}{2}dL_t^0(Y), \quad Y \geq 0 \quad (2)$$

where B is Brownian motion, σ and b are borel functions, $\forall a \in \mathbb{R}$, $L_t^a(Y)$ is the local time of Y at a . In the same subsection, we give also some explicit sufficient conditions which ensure the pathwise uniqueness for the SDE (2).

Subsection (4.2): We establish both the pathwise uniqueness and uniqueness in law for the SDE (1).

Subsection (4.3): Is dedicated to the pathwise uniqueness of SDE with a non sticky boundary condition. This result extends an early work of S. Manabe and T. Shiga [16].

Section (5): Prokaj in [26] has showed a recent result on the pathwise uniqueness of the so-called perturbed Tanaka equation:

$$Y_t = y + \int_0^t \text{sign}(Y_s) dM_t + \lambda N_t \quad (3)$$

In section 5, we use the local time technics introduced by Perkins [25] and further developed by LeGall [14], to provide a simple proof of a more general result of this type.

Subsection 5.1: We suggest some open problems .

1.1 Preliminaries

Throughout this paper we apply the usual conventions and currently standard notation of stochastic calculus associated with Itô integral. In particular, we denote by $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}; \mathbb{P})$ a filtered probability space, satisfying the usual conditions.

The notion of local time was introduced by P. Lévy for measuring the time spent by a diffusion process in the vicinity of a point. Following Ouknine (1991), we can distinguish three local times at a associated with the continuous semimartingale started at x :

- the right local time at a denoted $L_t^{a+}(X)$:

$$\frac{1}{2}L_t^{a+}(X) = (X_t - a)^+ - (x - a)^+ - \int_0^t 1_{\{X_s > a\}} dX_s$$

- the left local time at a denoted $L_t^{a-}(X)$:

$$\frac{1}{2}L_t^{a-}(X) = (X_t - a)^- - (x - a)^- - \int_0^t 1_{\{X_s < a\}} dX_s$$

- the symmetric local time at a denoted $L_t^a(X)$:

$$L_t^a(X) = (L_t^{a+}(X) + L_t^{a-}(X))/2$$

Proposition 1.1 (Occupation times formula) *For any bounded measurable function f and for all $t \geq 0$.*

$$\int_0^t f(X_s) d\langle X \rangle_s = \int_{\mathbb{R}} f(x) L_t^x(X) dx.$$

Proposition 1.2 *For any a , the measure $dL^a(X)$ is a.s. carried by the set $\{t \geq 0 : X_t = a\}$.*

Proposition 1.3 *There exists a modification of the random field $(L_t^a(X) ; a \in \mathbb{R}, t \geq 0)$ such that the map $(a, t) \longrightarrow L_t^a(X)$ is a.s. continuous in t and càdlàg in a . Moreover,*

$$L_t^a(X) - L_t^{a-}(X) = 2 \int_0^t 1_{\{X_s = 0\}} dX_s, \quad t \geq 0.$$

We refer the reader to the book of Protter [27] for a complete account on local times. Through this paper the indicator function is denoted 1 . We define \vee and \wedge through $a \vee b = \max(a, b)$ and $a \wedge b = \min(a, b)$. The positive and negative part is given respectively by $x^+ = x \vee 0$, $x^- = x \wedge 0$.

2 Continuous case

In this section we derive a generalization of comparison theorem for local times of semimartingales proved in [20] in the case of continuous semimartin-

gales. The proof used here does not need the upcrossing approximation of the local time process as in [20]. In order to give such an extension, we need first to recall the balayage formula.

At the beginning, let $X = (X_t, \ t \geq 0)$ be a continuous \mathcal{F}_t -semimartingale issued from zero. For every $t > 0$ we define

$$\gamma_t = \sup\{s \leq t : X_s = 0\},$$

with the convention $\sup(\emptyset) = 0$, hence in particular $\gamma_0 = 0$. The random variables γ_t are clearly not stopping times since they depend on the future.

Let us recall an important result of Azéma-Yor[1]:

Proposition 2.1 (*Balayage Formula*)

(i) *Let X be a continuous semimartingale, if k is a locally bounded predictable process, then*

$$k_{\gamma_t} X_t = k_0 X_0 + \int_0^t k_{\gamma_s} dX_s,$$

and therefore $(k_{\gamma_t} X_t)_{t \geq 0}$ is a continuous semimartingale. Moreover, if k is nonnegative then,

$$L_t^0(k_{\gamma} X) = \int_0^t k_s dL_s^0(X).$$

(ii) *If X is a local martingale. $k_{\gamma} X$ is also a local martingale and its local time at 0 is equal to*

$$\int_0^t k_s dL_s^0(X).$$

In the two following theorems we use balayage formula to give a clean proof of theorems originated from Nasyrov [18].

Theorem 2.1 (*The generalized Tanaka formula*) *For $t \geq 0$, $z > 0$, we have*

$$\frac{1}{2}(z \wedge L_t^0(X)) = 1_{\{L_t^0 \leq z\}} X_t^+ - X_0^+ - \int_0^t 1_{\{X_s \geq 0, L_s^0 \leq z\}} dX_s.$$

Proof. Let us set $k_t = 1_{\{L_t^0 \leq z\}}$,
Balayage formula yields,

$$k_{\gamma_t} \cdot X_t^+ = k_0 X_0^+ + \int_0^t k_{\gamma_s} dX_s^+$$

Using Tanaka's formula, this equality can be reexpressed as:

$$k_{\gamma_t} \cdot X_t^+ = X_0^+ + \int_0^t 1_{\{L_{\gamma_s}^0 \leq z\}} 1_{\{X_s \geq 0\}} dX_s + \frac{1}{2} \int_0^t 1_{\{L_{\gamma_s}^0 \leq z\}} dL_s^0(X).$$

Since the measure $dL^0(X)$ is carried by the set $\{s, X_s = 0\}$ and by the definition of γ_t , we see that $L_{\gamma_t}^0(X) = L_t^0(X)$, and then

$$1_{\{L_t^0 \leq z\}} X_t^+ = X_0^+ + \int_0^t 1_{\{X_s > 0, L_s^0 \leq z\}} dX_s + \frac{1}{2} (z \wedge L_t^0(X)).$$

Thus the result. ■

Remark 2.1 *It is obvious that $z = +\infty$ in precedent theorem, corresponds to the classical Tanaka formula.*

Theorem 2.2 *(The generalized Skorokhod equation) Let Φ be a locally integrable function. For $t \geq 0$,*

$$\Phi(L_t^0(X)) |X_t| = \Phi(0) |X_0| + \int_0^t \text{sgn}(X_s) \Phi(L_s^0(X)) dX_s + \int_0^{L_t^0(X)} \Phi(z) dz.$$

Moreover, if Φ is continuous and strictly positive function, we have

$$\int_0^{L_t^0(X)} \Phi(z) dz = - \min_{0 \leq s \leq t} \min \left(\int_0^s \text{sgn}(X_u) \Phi(L_u^0(X)) dX_u, 0 \right).$$

Proof. Let us set $k_t = \Phi(L_t^0(X))$ we denote γ_t the random variable

$$\gamma_t = \sup\{s < t, X_s = 0\}.$$

Balayage formula gives,

$$k_{\gamma_t}|X_t| = k_0|X_0| + \int_0^t k_{\gamma_s} d|X_s|.$$

If we apply Tanaka's formula on $|X|$, we get:

$$\begin{aligned} k_{\gamma_t}|X_t| &= k_0|X_0| + \int_0^t k_{\gamma_s} \operatorname{sgn}(X_s) dX_s + \int_0^t k_{\gamma_s} dL_s^0(X) \\ &= k_0|X_0| + \int_0^t \Phi(L_s^0(X)) \operatorname{sgn}(X_s) dX_s + \int_0^t \Phi(L_s^0(X)) dL_s^0(X) \end{aligned}$$

Since $(L_t^0(X), t \geq 0)$ is continuous nondecreasing process, the equality

$$\int_0^t \Phi(L_s^0(X)) dL_s^0(X) = \int_0^{L_t^0(X)} \Phi(z) dz,$$

holds for any $t > 0$. For more details see Nasyrov [18]. Thus,

$$k_{\gamma_t}|X_t| = k_0|X_0| + \int_0^t \Phi(L_s^0(X)) \operatorname{sgn}(X_s) dX_s + \int_0^{L_t^0(X)} \Phi(z) dz$$

, whence the desired result. ■

Now, the inequality between local times combined with the balayage formula allow us to establish our main result:

Theorem 2.3 *Let X and Y be two continuous semimartingales such that*

(1) $\{s \geq 0 / X_s = 0\} \subset \{s \geq 0 / Y_s = 0\}$ *and,*

(2) $X_s^+ \leq Y_s^+$,

then the measure $dL_t^0(X)$ is absolutely continuous with respect to the measure $dL_t^0(Y)$:

$$dL_t^0(X) \ll dL_t^0(Y),$$

and there exist a predictable process $(\theta_s)_{s \geq 0}$, $\theta_s \in [0, 1]$ such that

$$L_t^0(X) = \int_0^t \theta_s dL_s^0(Y),$$

Proof. We recall first that $L_t^0(Z) = L_t^0(Z^+)$ for every semimartingale Z . Hence it is enough to consider the case where X and Y are non-negatives

semimartingales.

Now, let us remark that if

$$\{s \geq 0 / X_s = 0\} \subset \{s \geq 0 / Y_s = 0\},$$

then,

$$\gamma_t^X = \gamma_t^Y = \gamma.$$

Let k be a predictable, positive and locally bounded process. Under the assumptions stated above, the processes

$$(\tilde{X}_t)_{t \geq 0} = (k_{\gamma_t} \cdot X_t)_{t \geq 0} \quad \text{and} \quad (\tilde{Y}_t)_{t \geq 0} = (k_{\gamma_t} \cdot Y_t)_{t \geq 0}$$

satisfy

$$\{s \geq 0 / \tilde{X}_s = 0\} \subset \{s \geq 0 / \tilde{Y}_s = 0\}.$$

It is clear that:

$$\tilde{X}_t \leq \tilde{Y}_t.$$

Following Ouknine [20], under this hypothesis,

$$L_t^0(\tilde{X}) \ll L_t^0(\tilde{Y}).$$

On the other hand, by the balayage formula:

$$L_t^0(\tilde{X}) = \int_0^t k_s dL_s^0(X) \quad \text{and} \quad L_t^0(\tilde{Y}) = \int_0^t k_s dL_s^0(Y).$$

Consequently,

$$\int_0^\cdot k_s dL_s^0(X) \ll \int_0^\cdot k_s dL_s^0(Y).$$

Hence,

$$dL_t^0(X) \ll dL_t^0(Y).$$

Therefore, by Radon-Nikodym theorem, there exists a predictable process θ , $\theta \in [0, 1]$ such that

$$L_t^0(X) = \int_0^t \theta_s dL_s^0(Y).$$

■

Remark 2.2 The Radon-Nikodym derivative of $dL_t^0(X)$ with respect to the measure $dL_t^0(Y)$ is precisely,

$$\theta_{\gamma_t} = \liminf_{\varepsilon \searrow 0} \frac{X_{\gamma_t + \varepsilon}}{Y_{\gamma_t + \varepsilon}}.$$

Remark 2.3 If there exists a continuous semimartingale ξ such that $X = \xi.Y$, the formula can be derived by a result of [19] giving the local time of product of two continuous semimartingales. In fact,

$$\begin{aligned} L_t^0(X) &= L_t^0(\xi.Y) \\ &= \int_0^t \xi_s^+ dL_s^0(Y) + \int_0^t Y_s^+ dL_s^0(\xi). \end{aligned}$$

The measure $dL_s^0(\xi)$ is carried by the set $\{s, \xi_s = 0\} \subset \{s, X_s = 0\}$. According to the assumption (1),

$$\{s, \xi_s = 0\} \subset \{s, Y_s = 0\}.$$

Consequently,

$$L_t^0(X) = \int_0^t \xi_s^+ dL_s^0(Y).$$

Hence de desired result.

Now, we show that The absolute continuity of the measure $dL_s^0(X)$ with respect to the measure $dL_s^0(Y)$ is localizable.

Proposition 2.2 Let A denote the set $\{\forall s \in [0, T] \quad 0 \leq X_s \leq Y_s\}$. If

$$\{s \geq 0 / X_s = 0\} \subset \{s \geq 0 / Y_s = 0\},$$

then,

$$dL_s^0(X) \ll dL_s^0(Y) \quad \text{on the set } A$$

Proof. Let X and Y be two continuous semimartingales. Let $\epsilon > 0$, $0 < t \leq T$. We denote $M_t^X([0, \epsilon])$ the number of upcrossings of ϵ by X on the interval $[0, t]$, (recall that X_s is said to have an upcrossing of ϵ at $s_0 > 0$ if for some $x > 0$, $X_s \leq \epsilon$ in $(s_0 - x, s_0)$ and $X_s \geq \epsilon$ in $(s_0, s_0 + x)$). On the set A we have

$$M_t^X([0, \epsilon]) \leq M_t^Y([0, \epsilon]);$$

by multiplying this inequality by 2ϵ , we get by Lévy result which says that $L_t^0(X) = \lim_{\epsilon \rightarrow 0} 2\epsilon M_t^X([0, \epsilon])$

$$L_t^0(X) \leq L_t^0(Y) \quad \text{on the set } A$$

To prove that $dL_t^0(X) \ll dL_t^0(Y)$ on the set A , we proceed by the same way as theorem 2.3 by using the balayage formula. ■

The purpose of this paragraph, it to show that under suitable but weak conditions, comparison result for local times can be proved. The idea of the proof is the same as in [20].

Let X be a continuous semimartingale null in 0. We set

$$Z(X) := \{s \leq T; X_s = 0\}$$

So,

$$Z(X)^c := \cup_{n \geq 1}]g_n, d_n[$$

where $]g_n, d_n[$ is the n^e interval of excursion around 0.

Set

$$M_n^X(t) = \sup_{s \in]g_n, d_n[\cap [0, t]} (X_s^+)$$

Proposition 2.3 *Let X and Y be two continuous semimartingales such that:*

1. $Z(X) = Z(Y)$,
2. $M_n^X(t) \leq M_n^Y(t), \forall n \in \mathbb{N}, \forall t \leq T$.

Then,

$$dL_t^0(X) \ll dL_t^0(Y).$$

Proof. Let X and Y be two continuous semimartingales. let $\epsilon > 0$, $0 < t \leq T$. We denote $M_t^X([0, \epsilon])$ the number of upcrossings of ϵ by X on the interval $[0, t]$. If $M_n^X(t) \leq M_n^Y(t)$ for all $n \in \mathbb{N}$, then we have:

$$M_t^X([0, \epsilon]) \leq M_t^Y([0, \epsilon]);$$

by multiplying this inequality by 2ϵ , we get by Lévy result [?]

$$L_t^0(X) \leq L_t^0(Y).$$

To prove that $dL_t^0(X) \ll dL_t^0(Y)$, we proceed by the same way as theorem (2.3) by using the balayage formula. ■

Corollary 1 *Let $X = (X_1, \dots, X_n)$ be a n -dimensional semimartingale, N_1 and N_2 be norms on \mathbb{R}^n such that $N_1 \leq N_2$. Then, $dL_t^0(N_1(X)) \leq dL_t^0(N_2(X))$.*

3 Càdlàg case

In [7], F. Coquet and Y. Ouknine give a new comparison theorem for local times concerning essentially càdlàg semimartingales. In this section, we focus our attention to extend this result to the measures associated to this local times.

We first introduce some notations which will be used in this section.

Let M be a random predictable set, closed on the right containing $\{0\}$. For $t \geq 0$, we define

$$\gamma_t = \sup\{s \leq t, s \in M\}.$$

Before we give statements of our main results, we recall this definition:

Definition 3.1 *Let Z be a semimartingale, we define the local time of Z , at 0 as the continuous adapted increasing process $L^0(Z)$ satisfying*

$$|Z_t| = |Z_0| + \int_0^t \text{sgn}(Z_{s-}) dZ_s + 2 \sum_{0 < s \leq t} [Z_s^- 1_{\{Z_{s-} > 0\}} + Z_s^+ 1_{\{Z_{s-} \leq 0\}}] + L_t^0(Z),$$

where we set $\text{sgn}(0) = -1$.

The balayage formula for càdlàg semimartingale is essential for the proof of our main result.

Proposition 3.1 *Let $(X_t)_{t \geq 0}$ be a right continuous process such that X is null on the M , and k be bounded and predictable process, $(k_{\gamma_t} X_t)_{t \geq 0}$ is a right continuous semimartingale and*

$$k_{\gamma_t} X_t = k_0 X_0 + \int_0^t k_{\gamma_s} dX_s.$$

Moreover,

$$L_t^0(k_\gamma.X) = \int_0^t |k_s| dL_s^0(X).$$

Owing to Tanaka formula for càdlàg semimartingales, it is natural to establish our main result in discontinuous case under new hypothesis.

Theorem 3.1 *Let X and Y be two càdlàg semimartingales such that*

- (1) $\{X = X_- = 0\} \subset \{Y = Y_- = 0\}$,
- (2) $X^+ \leq Y^+$.

Then,

$$dL_t^0(X) \ll dL_t^0(Y) \quad \text{for all } t > 0.$$

And there exists a predictable process $(\theta_s)_{s \geq 0}$, $\theta_s \in [0, 1]$ such that:

$$L_t^0(X) = \int_0^t \theta_s dL_s^0(Y)$$

Proof of theorem. Without loss of generality, it suffices to consider the case where X and Y are non-negative semimartingales. Let $CP(X)$ denotes the continuous part of a process X . It is well known that

$$\begin{aligned} \frac{1}{2}L_t^0(X) &= CP \left[\int_0^t 1_{\{X_{s-}=0\}} dX_s \right] \\ &= CP \left[\int_0^t 1_{\{X_{s-}=X_s=0\}} dX_s \right] \end{aligned}$$

the process $V_t = \int_0^t 1_{\{X_{s-}=0\}} dX_s$ is of bounded variation, so this integral

$$\int_0^t 1_{\{X_{s-}=X_s=0\}} dX_s$$

is well defined. And we have:

$$\frac{1}{2}L_t^0(X) = CP \left[\int_0^t 1_{\{X_s=X_{s-}=0\}} dY_s - \int_0^t 1_{\{X_s=X_{s-}=0\}} d(Y_s - X_s) \right],$$

see [19] for more details. Using this assumption:

$$\{X = X_- = 0\} \subset \{Y = Y_- = 0\},$$

we obtain this inequality:

$$\frac{1}{2}L_t^0(X) \leq CP \left[\int_0^t 1_{\{Y_s=Y_{s-}=0\}} dY_s \right] - CP \left[\int_0^t 1_{\{X_s=X_{s-}=0\}} d(Y-X)_s \right],$$

and the same assumption allows us to write:

$$\begin{aligned} \int_0^t 1_{\{X_s=X_{s-}=0\}} d(Y-X)_s &= \int_0^t 1_{\{X_s=X_{s-}=0\}} 1_{\{Y_{s-}-X_{s-}=0\}} d(Y-X)_s \\ &= \frac{1}{2} \int_0^t 1_{\{X_{s-}=X_s=0\}} dL_s^0(Y-X) + \int_0^t 1_{\{X_{s-}=X_s=0\}} dA_s \end{aligned}$$

where

$$A_t = \sum_{s \leq t} 1_{\{Y_{s-} \leq X_{s-}\}} (Y_s - X_s)^+ + 1_{\{Y_{s-} > X_{s-}\}} (Y_s - X_s)^-.$$

Therefore, $\int_0^t 1_{\{X_{s-}=X_s=0\}} d(Y-X)_s \geq 0$, this clearly forces:

$$\frac{1}{2}L_t^0(X) \leq CP \left(\int_0^t 1_{\{Y_{s-}=0\}} dY_s \right),$$

which establishes the following formula:

$$L_t^0(X) \leq L_t^0(Y).$$

To complete the proof, we set

$$M = \{X_{s-} = X_s = 0\}.$$

Using the balayage formula we can define the processes:

$$\tilde{X}_t = k_{\gamma_t} X_t = \int_0^t k_{\gamma_s} dX_s \quad \text{and} \quad \tilde{Y}_t = k_{\gamma_t} Y_t = \int_0^t k_{\gamma_s} dY_s$$

which are càdlàg semimartingales satisfying the assumptions (1) and (2). Consequently,

$$L_t^0(\tilde{X}) \leq L_t^0(\tilde{Y}).$$

Therefore, we obtain

$$\int_0^t k_s dL_s^0(X) \leq \int_0^t k_s dL_s^0(Y),$$

Thus, the result follows by a particular choice of the process k . ■

4 Application to SDE

4.1 Pathwise uniqueness for some reflected SDE's

We deal this section by recalling briefly some type of uniqueness of solution to stochastic differential equation. For more details about this notion, we refer the reader to [28].

We consider the following stochastic differential equation:

$$X_t = X_0 + \int_0^t \sigma(X_s) dB_s + \int_0^t b(X_s) ds, \quad e(\sigma, b)$$

where X_0 is a random variable and b, σ are two Borel functions.

We suppose that, $\forall t \geq 0$,

$$\int_0^t \sigma^2(X_s) ds < +\infty, \quad \int_0^t |b(X_s)| ds < +\infty.$$

We associate to $e(\sigma, b)$ another equation $e'(\sigma, b)$ by addition of local time component:

$$\begin{cases} dY_t = \sigma(Y_t) dB_t + b(Y_t) dt + \frac{1}{2} dL_t^0(Y) \\ Y_t \geq 0 \end{cases} \quad e'(\sigma, b)$$

Definition 4.1 1. A solution of the SDE $e(\sigma, b)$ (or $e'(\sigma, b)$) is a pair (X, B) of adapted processes defined on a probability space $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ such that

- B is a standard \mathcal{F} -Brownian motion in \mathbb{R} ,
- X satisfies $e(\sigma, b)$ or $e'(\sigma, b)$ and the conditions above.

2. A solution X of the SDE $e(\sigma, b)$ or $e'(\sigma, b)$ is said to be trivial if

$$\mathbb{P}(\{X_t = X_0, \forall t \geq 0\}) = 1.$$

In Proposition 3.2, Chap. IX of Ref. [28], it is shown that if uniqueness in law holds for $e(\sigma, b)$ and if the local time $L^0(X^1 - X^2) = 0$ for any pair of solutions such that $X_0^1 = X_0^2$ a.s., then the pathwise uniqueness holds for $e(\sigma, b)$.

The analogue of this result for the SDE $e'(\sigma, b)$ is given by this theorem:

Theorem 4.1 *The two following properties are equivalent:*

1. *There is pathwise uniqueness for equation $e'(\sigma, b)$.*
2. *There is uniqueness in law for equation $e'(\sigma, b)$ and whenever (X, B) and (Y, B) are two solutions such that $X_0 = Y_0$ a.s., $dL_s^0(X - Y)$ is carried by the set $\{s, X_s = Y_s = 0\}$.*

Proof.

(1) \Rightarrow (2). Trivial.

(2) \Rightarrow (1). We will prove that whenever X and Y are solutions of $e'(\sigma, b)$, $X \vee Y$ and $X \wedge Y$ are also solutions.

By Tanaka's formula one has:

$$\begin{aligned} (X_t \vee Y_t) &= Y_t + (X_t - Y_t)^+ \\ &= Y_t + \left(\int_0^t 1_{\{X_s > Y_s\}} d(X_s - Y_s) + \frac{1}{2} L_t^0(X - Y) \right) \\ &= \int_0^t \sigma(X_s \vee Y_s) dB_s + \int_0^t b(X_s \vee Y_s) ds \\ &\quad + \frac{1}{2} \int_0^t 1_{\{X_s > Y_s\}} dL_s^0(X) + \frac{1}{2} \int_0^t 1_{\{X_s \leq Y_s\}} dL_s^0(Y) + \frac{1}{2} L_t^0(X - Y). \end{aligned}$$

Observe that by a support argument for local time we have: $\forall t \geq 0$,

$$\int_0^t 1_{\{X_s > Y_s\}} dL_s^0(X) = \int_0^t 1_{\{0 > Y_s\}} dL_s^0(X) \quad \text{and} \quad \int_0^t 1_{\{X_s \leq Y_s\}} dL_s^0(Y) = \int_0^t 1_{\{X_s \leq 0\}} dL_s^0(Y).$$

Since the measure $dL_s^0(X - Y)$ is carried by the set $\{s, X_s = Y_s = 0\}$, $L_t^0(X - Y)$ can be expressed as:

$$L_t^0(X - Y) = \int_0^t 1_{\{X_s=Y_s=0\}} dL_s^0(X - Y).$$

Now from a result by Ouknine and Rutkowski [23], in which they prove the local time of sup of two semimartingales:

$$L_t^0(X \vee Y) = \int_0^t 1_{\{X_s < 0\}} dL_s^0(Y) + \int_0^t 1_{\{Y_s \leq 0\}} dL_s^0(X) + \int_0^t 1_{\{X_s=Y_s=0\}} dL_s^0(Y - X).$$

Hence $X \vee Y$ is a solution to $e'(\sigma, b)$. Similarly we can show that $X \wedge Y$ is a solution as well. As X and Y have integrable paths on finite time interval, then

$$\mathbb{E}[|X_t - Y_t|] = \mathbb{E}[X_t \vee Y_t] - \mathbb{E}[X_t \wedge Y_t]$$

and by uniqueness in law, we obtain:

$$\mathbb{E}[|X_t - Y_t|] = 0,$$

then, X and Y are indistinguishable. ■

In the next step we show that under some hypothesis on the diffusion coefficient σ , we state some sufficient conditions for the pathwise uniqueness of the equation $e'(\sigma, b)$. Let us first recall an important result of Ouknine [22]:

Proposition 4.1 *Suppose σ and b are two bounded borel functions. If σ verifies the following condition:*

1. $|\sigma| \geq \varepsilon > 0$
2. *there exists $n \in 2\mathbb{N}$ such that*

$$|x^n \sigma(x) - y^n \sigma(y)|^2 \leq \rho(|h(x) - h(y)|), \text{ for all } x, y \in \mathbb{R}$$

where $\rho : [0, \infty) \rightarrow [0, \infty)$ is continuous and nondecreasing, $\rho(0) = 0$,

$\rho(x) > 0$ for $x > 0$, and $\rho(\alpha x) \leq \alpha \rho(x) \quad \forall \alpha > 1, \forall x > 0$, and

$$\int_0^\epsilon \frac{du}{\rho(u)} = +\infty \text{ for some } \epsilon > 0,$$

then we have pathwise uniqueness for equation $e'(\sigma, b)$.

Remark 4.1 In 1988, to prove the pathwise uniqueness of solutions of $e'(\sigma, b)$, Ouknine [20] tried to prove that $L_t(X - Y) = 0$ whenever X and Y are solutions. In this time, he has just proved that

$$\int_0^t (X_s^{2p} + Y_s^{2p}) dL_s^0(X - Y) = 0,$$

which with the Theorem 4.1 is sufficiently enough to prove the pathwise uniqueness of $e'(\sigma, b)$.

We can prove the same result as above but under weaker conditions.

Proposition 4.2 If σ verify the following conditions

1. $|\sigma| \geq \varepsilon > 0$,
2. there exists $n \in \mathbb{N}$, $c > 0$ such that

$$|x^{n-1}\sigma(x) - y^{n-1}\sigma(y)|^2 \leq c\rho(|x^n - y^n|) \quad \forall x, y \in \mathbb{R},$$

where $\rho : [0, \infty) \rightarrow [0, \infty)$ is continuous and nondecreasing, $\rho(0) = 0$, $\rho(x) > 0$ for $x > 0$, and

$$\int_{0+}^\epsilon \frac{du}{\rho(u)} = +\infty \quad \text{for some } \epsilon > 0,$$

then we have pathwise uniqueness for equation $e'(\sigma, b)$.

Proof. Let X and Y be two solutions of $e'(\sigma, b)$ defined on a common probability basis with respect to same Brownian motion such that $X_0 = Y_0$. The hypothesis (1) is sufficient for uniqueness in law.

The task now is to prove that the measure $dL_s(X - Y)$ is carried by the

set $\{s, X_s = Y_s = 0\}$. Let us show first that $L_t^0(X^n - Y^n) = 0$.

Applying the assumption (2) combined with Itô's formula, we obtain:

$$\begin{aligned} \int_0^t 1_{\{X_s^n - Y_s^n > 0\}} \frac{d\langle X^n - Y^n \rangle_s}{\rho(X_s^n - Y_s^n)} &\leq n^2 \int_0^t \frac{(X_s^{n-1} \sigma(X_s) - Y_s^{n-1} \sigma(Y_s))^2}{\rho(X_s^n - Y_s^n)} 1_{\{X_s^n - Y_s^n > 0\}} ds \\ &\leq n^2 ct. \end{aligned}$$

Hence, by the occupation times formula, we get:

$$\int_0^t 1_{\{X_s^n - Y_s^n > 0\}} \frac{d\langle X^n - Y^n \rangle_s}{\rho(X_s^n - Y_s^n)} = \int_{0+} \frac{da}{\rho(a)} L_t^a(X^n - Y^n) < \infty.$$

Thus with the help of the right continuity of $a \rightarrow L_t^a(X^n - Y^n)$ and the condition $\int_{0+}^\epsilon \frac{du}{\rho(u)} = \infty$ for some $\epsilon > 0$, $L_t^0(X^n - Y^n) = 0$.

On the other hand, $X^n - Y^n$ can be written as :

$$X^n - Y^n = (X - Y)P(X, Y),$$

with $P(X, Y) = \sum_{k=0}^{n-1} X^k Y^{n-k-1}$. So,

$$L_t^0(X^n - Y^n) = L_t^0[(X - Y)P(X, Y)] \quad \forall n \in \mathbb{N}^*$$

Using the formula giving the local time of product of two semimartingales which goes back to Ouknine[19], we obtain:

$$\begin{aligned} L_t^0(X^n - Y^n) &= L_t^0[(X - Y)P(X, Y)] \\ &= \int_0^t (X_s - Y_s)^+ dL_s^0(P(X, Y)) + \int_0^t P(X_s, Y_s) dL_s^0(X - Y). \end{aligned} \tag{4}$$

But $\int_0^t (X_s - Y_s) dL_s^0(P(X, Y)) = 0$, since the measure $dL_s^0(P(X, Y))$ is carried by the set $\{s, P(X, Y) = 0\} = \{s, X_s = Y_s = 0\}$, the last equality is due to the positivity of X and Y .

(4) now becomes :

$$\begin{aligned} \int_0^t P(X_s, Y_s) dL_s^0(X - Y) &= n \int_0^t X^{n-1} dL_s^0(X - Y) \\ &= 0. \end{aligned}$$

Finally, owing to the positivity of X , the measure $dL_s^0(X - Y)$ is carried by the set of $\{s, X_s = Y_s = 0\}$. ■

Proposition 4.3 *Let X and Y be two solutions of equation $e'(\sigma, b)$. If σ satisfies the assumptions of the previous proposition, $Z := X - Y$, and $Z' := X^n - Y^n$. Then,*

$$dL_t(Z) \perp dL_t(Z').$$

Proof. By the same argument in the proof of Proposition 4.2,

$$\int_0^t 1_{\{X_s=Y_s=0\}} dL_s^0(X^n - Y^n) = 0.$$

But owing to assumptions satisfied by σ , the measure $dL_s^0(X - Y)$ is carried by the set $\{s, X_s = Y_s = 0\}$ which completes the proof. ■

By the same lines as the previous proof of proposition 4.2 we have the following more general setting:

Proposition 4.4 *There is pathwise uniqueness for equation $e'(\sigma, b)$ if the two following conditions hold:*

- (i) *There is uniqueness in law for equation $e'(\sigma, b)$,*
- (ii) *σ verifies the following condition:*

$$|f'(x)\sigma(x) - f'(y)\sigma(y)|^2 \leq c\rho(|f(x) - f(y)|) \quad \text{for all } x, y \in \mathbb{R},$$

where $f : \mathbb{R} \rightarrow \mathbb{R}$ is strictly increasing, continuously differentiable, and it is a difference of two convex functions, its derivative f' satisfies:

$$f'(x) = 0 \Leftrightarrow x = 0,$$

and $\rho : [0, \infty) \rightarrow [0, \infty)$ is continuous and nondecreasing, $\rho(0) = 0$, $\rho(x) > 0$ for $x > 0$, such that:

$$\int_0^\epsilon \frac{du}{\rho(u)} = \infty \quad \text{for some } \epsilon > 0.$$

The following corollary is essential for the proof of Proposition 4.4 (see Ouknine and Rutkowski [23]).

Corollary 2 Suppose that the function $g : \mathbb{R} \rightarrow \mathbb{R}$ is strictly increasing, continuously differentiable, and g is a difference of two convex functions, then we have:

$$L_t^0(g(X) - g(Y)) = \int_0^t g'(X_s) dL_s^0(X - Y).$$

Proof of Proposition 4.4. Let (X, B) and (Y, B) two solutions of the SDE $e'(\sigma, b)$ such that $X_0 = Y_0$.

Let assumptions (i), (ii) hold. Then:

$$\begin{aligned} \int_0^t 1_{\{f(X_s) - f(Y_s) > 0\}} \frac{d\langle f(X) - f(Y) \rangle_s}{\rho(f(X_s) - f(Y_s))} &\leq \int_0^t 1_{\{f(X_s) - f(Y_s) > 0\}} \frac{(f'(X_s)\sigma(X_s) - f'(Y_s)\sigma(Y_s))^2}{\rho(f(X_s) - f(Y_s))} ds \\ &\leq ct \end{aligned}$$

By the same arguments as in the last paragraph of the proof of the Proposition (4.2):

$$L_t^0(f(X) - f(Y)) = 0.$$

On account of Corollary(2), we have:

$$\int_0^t f'(X_s) dL_s^0(X - Y) = 0.$$

Since $f'(x) = 0 \Leftrightarrow x = 0$, we can conclude that the measure $dL_t^0(X - Y)$ is carried by the set $\{t/ X_t = Y_t = 0\}$. ■

Remark 4.2 We insist that in case (ii) σ does not depend on s as in Proposition (4.2).

Now, we introduce the definition of $(LT)_+$ condition.

Definition 4.2 σ satisfies $(LT)_+$ if whenever V^1 and V^2 are continuous adapted processes of bounded variation on some $(\Omega, \mathcal{F}, \mathbb{P})$ and X^i ($i = 1, 2$) are **positive** adapted solutions of

$$X_t^i = x_i + \int_0^t \sigma(X_s^i) dB_s + V_s^i \quad i = 1, 2$$

$(x_1, x_2 \in \mathbb{R})$, then $L_t^0(X^1 - X^2) = 0$ for all $t \geq 0$.

Under $(LT)_+$ condition the pathwise uniqueness fails for solutions to the SDE $e(\sigma, b)$. However we can show that this uniqueness is valid for its absolute value.

Proposition 4.5 *Suppose that b is an odd bounded measurable function on \mathbb{R} . Suppose that σ is an odd bounded measurable function on \mathbb{R} with a^{-2} locally integrable on \mathbb{R} and satisfies $(LT)_+$ condition. Then for any two weak solutions X and Y to this SDE:*

$$dX_t = \sigma(X_t) dB_t + b(X_t) dt \quad (5)$$

with a common Brownian motion on a common probability space with $X_0 = Y_0$. We have the pathwise uniqueness for this reflected stochastic differential equation:

$$d|X_t| = \sigma(|X_t|) dB_t + b(|X_t|) dt + \frac{1}{2} dL_t^0(|X|)$$

where $L_t^0(X)$ is the local time of X .

Proof. Suppose that σ and b satisfy the assumptions of the proposition. Let X and Y two weak solutions to (5) with a common Brownian motion on a common probability space and $X_0 = Y_0$. X satisfies (5) and σ is odd, then by Tanaka's formula

$$d|X_t| = \sigma(|X_t|) dB_t + b(|X_t|) dt + \frac{1}{2} dL_t^0(|X|)$$

Similarly $|Y_t|$ satisfies equation with Y_t in the place of $|X|$. By $(LT)_+$ condition we get the desired result. ■

It is shown by Barlow [2] that

$$dX_t = (a1_{\{X_t > 0\}} - b1_{\{X_t \leq 0\}}) dB_t$$

may not have any strong solutions if a and b are strictly positive. Then the pathwise uniqueness fails. However, we will show the pathwise uniqueness for $\tilde{X} = \frac{1}{a}X^+ + \frac{1}{b}X^-$.

Proposition 4.6 *Let X and Y be two weak solutions to this SDE:*

$$dX_t = (a1_{\{X_t > 0\}} - b1_{\{X_t \leq 0\}}) dB_t \quad (6)$$

with a common Brownian motion on a common probability space such that $X_0 = Y_0$. then we have:

$$\mathbb{P} \left(\frac{1}{a}X_t^+ + \frac{1}{b}X_t^- = \frac{1}{a}Y_t^+ + \frac{1}{b}Y_t^- \text{ for all } t \geq 0 \right) = 1$$

Proof. Let (X, B) and (Y, B) be two weak solutions to (6) driving by the same Brownian motion B on a common probability space, $X_0 = Y_0$. Let $\alpha > 0, \beta > 0$. We set $\phi(X_t) = \alpha X_t^+ + \beta X_t^-$. By Itô's formula we have

$$\phi(X_t) = \int_0^t (\alpha a 1_{\{X_s > 0\}} + \beta b 1_{\{X_s \leq 0\}}) dB_s + \frac{1}{2}(\alpha + \beta)L_t^0(X) \quad (7)$$

Tanaka's formula gives

$$\begin{aligned} L_t^0(\phi(X)) &= 2 \int_0^t 1_{\{\phi(X_s)=0\}} d\phi(X_s) \\ &= 2 \int_0^t \alpha 1_{\{X_s=0\}} dX_s^+ + 2 \int_0^t \beta 1_{\{X_s=0\}} dX_s^- \\ &= (\alpha + \beta)L_t^0(X). \end{aligned}$$

Setting $\alpha = \frac{1}{a}, \beta = \frac{1}{b}$, and substituting in (7), we get

$$\phi(X_t) = B_t + \frac{1}{2}L_t^0(\phi(X)). \quad (8)$$

Thus, denoting by $\phi(Y)$ the process obtained in the same way from Y , and so $\phi(Y)$ satisfies equation (8) with $\phi(Y)$ in place of $\phi(X)$. Since $\frac{1}{2} < 1$, the equation (8) has unique strong solution (see [11]). Consequently,

$$\mathbb{P}(\phi(X_t) = \phi(Y_t) \text{ for all } t \geq 0) = 1$$

. ■

4.2 SDE with local time

In this subsection, the main subject of this part, the stochastic differential equations involving local times of unknown process, is treated. A special case of such equations was introduced by Harisson and shepp [11], they have established that the equation:

$$X_t = B_t + \beta L_t^0(X),$$

has no solution if $|\beta| > 1$, and has a unique strong solution when $|\beta| \leq 1$. Next Le Gall [15] has studied equation of a form :

$$X_t = X_0 + \int_0^t \sigma_s(X_s) dB_s + \int_{\mathbb{R}} L_t^a(X) \nu(da) \quad (9)$$

where ν stands for a finite signed measure in \mathbb{R} , and the diffusion coefficient is assumed to be strictly positive function of finite variation.

We shall prove pathwise uniqueness of solutions to more general equation, corresponding to non Markovian type:

$$X_t = X_0 + \int_0^t \sigma_s(X_s, B_s) dB_s + \int_{\mathbb{R}} L_t^a(X) \nu(da) \quad (10)$$

where

- (i) L^a is the local time at a for the semimartingale X and ν is a σ -finite measure with $|\nu|(\{a\}) < 1, \forall a \in \mathbb{R}$,
- (ii) B is a standard Wiener process,
- (iii) the mapping function $\sigma : \mathbb{R}^+ \times C(\mathbb{R}^+, \mathbb{R}) \times C(\mathbb{R}^+, \mathbb{R}) \rightarrow \mathbb{R}$ is measurable and adapted to the filtration $(\mathcal{C}_t)_{t \geq 0}$.

Remark 4.3 If $\sigma \equiv 1$ and ν is concentrated in the point 0 with $|\nu|(\{0\}) < 1$, equation (10) describes the skew Brownian motion which was treated by several authors, e.g. Harisson and Shepp (1981), Le Gall (1982), Ouknine (1991).

Remark 4.4 In a certain sense, the jump condition:

$$|\nu|(\{a\}) < 1$$

is a natural restriction since there is, in general, no solution of equation (10) if $\nu(\{a\}) < -1$ for some $a \in \mathbb{R}$ (cf. [11]).

Since we are interested by stochastic differential equations with a general mapping functions σ . We shall generalize the definition of (LT) condition introduced in Barlow and Perkins [2].

Definition 4.3 σ satisfies (LT) if whenever V^1 and V^2 are continuous adapted processes of bounded variation on some $(\Omega, \mathcal{F}, \mathbb{P})$ and X^i ($i = 1, 2$) are adapted solutions of

$$X_t^i = x_0 + \int_0^t \sigma_s(X^i, B) dB_s + V_s^i \quad i = 1, 2$$

where $\sigma : \mathbb{R}^+ \times C(\mathbb{R}^+, \mathbb{R}) \times C(\mathbb{R}^+, \mathbb{R}) \rightarrow \mathbb{R}$ is measurable and adapted to the filtration $(\mathcal{C}_t)_{t \geq 0}$. then $L_t^0(X^1 - X^2) = 0$ for all $t \geq 0$.

The (LT) condition will help us to get pathwise uniqueness for the SDE (10).

Theorem 4.2 If σ satisfies (LT) condition, we have the following properties:

- (i) The solution to (10) is pathwise unique.
- (ii) There is uniqueness in law for equation (10).

Proof.

(i) Let (X, B) and (X', B) be two solutions of the SDE (10) with the same Brownian motion B on the same probability space.

After localization, one can suppose that X is bounded and ν is σ -finite measure because $a \rightarrow L_t^a(X)$ is compactly supported. The main tool to do this proof is the following transformation. Let us set

$$f_\nu(y) = e^{-2\nu^c]0, y]} \prod_{z \leq y} \left(\frac{1 - \nu\{z\}}{1 + \nu\{z\}} \right),$$

where ν^c is the continuous part of the measure ν . It is well known that the function f_ν satisfies

$$0 < m \leq f_\nu(x) \leq M \quad \forall x \in \mathbb{R}$$

for some constants m, M . We set

$$F_\nu(x) = \int_0^x e^{-2\nu^c]0,y]} \prod_{z \leq y} \left(\frac{1 - \nu\{z\}}{1 + \nu\{z\}} \right) dy,$$

hence the function F_ν is increasing, bijective and is a difference of two convex functions. We set

$$Y = F_\nu(X), \quad \text{and} \quad Y' = F_\nu(X').$$

By application of Itô-Tanaka's formula one has:

$$Y_t = F_\nu(x) + \int_0^t \tilde{\sigma}_s(X, B) dB_s,$$

and,

$$Y'_t = F_\nu(x) + \int_0^t \tilde{\sigma}_s(X', B) dB_s.$$

where $\tilde{\sigma}_s(X, B) = f_\nu(X_s)\sigma_s(X, B)$.

Plainly,

$$m(X_t - X'_t)^+ \leq (Y_t - Y'_t)^+ \leq M(X_t - X'_t)^+,$$

By comparison theorem for local times (see [19]):

$$L_t^0(Y - Y') \leq ML_t^0(X - X').$$

As σ verifies (LT) condition, we have

$$L_t^0(X - X') = 0.$$

It follows easily that $L_t^0(Y - Y') = 0$. Moreover, $Y - Y'$ is continuous martingale and $Y_0 - Y'_0 = 0$, and so $Y = Y'$. This implies that the paths of X are uniquely determined.

(ii) To prove the uniqueness in law, we use the theorem of Engelbert-Yamada-Watanabe [8] which shows that the pathwise uniqueness implies uniqueness in law. ■

As an immediate application of the previous theorem, we have:

Corollary 3 Let b and σ be two measurable functions on \mathbb{R} . By N_σ we denote the set $N_\sigma = \{x : \sigma(x) = 0\}$ of zero of σ .

If

1. σ satisfies (LT) condition,

2. $\frac{b}{\sigma^2} 1_{N_\sigma^c} \in L^1_{loc}(\mathbb{R})$.

Then the pathwise uniqueness holds for:

$$\begin{cases} dX_t = \sigma(X_t) dB_t + b(X_t) dt \\ \int_0^\infty 1_{N_\sigma}(X_s) ds = 0 \end{cases} \quad (11)$$

Proof. Let X be a solution of (11) for which the time spent at N_σ has Lebesgue measure 0.

The equation (11) can be written as:

$$X_t = X_0 + \int_0^t \sigma(X_s) dB_s + \int_0^t b(X_s) 1_{N_\sigma}(X_s) ds + \int_0^t \frac{b(X_s)}{\sigma^2(X_s)} \cdot \sigma^2(X_s) 1_{N_\sigma^c}(X_s) ds$$

By (LT) condition

$$\int_0^t b(X_s) 1_{N_\sigma}(X_s) ds = 0,$$

then using occupation times formula, we may reexpress X_t as follows :

$$\begin{aligned} X_t &= X_0 + \int_0^t \sigma(X_s) dB_s + \int_0^t \frac{b(X_s)}{\sigma^2(X_s)} 1_{N_\sigma^c}(X_s) d\langle X \rangle_s \\ &= X_0 + \int_0^t \sigma(X_s) dB_s + \int_{\mathbb{R}} L_t^x(X) \mu(dx) \end{aligned}$$

with $L_t^x(X)$ is the local time of X , $\mu(dx) = \frac{b(x)}{\sigma^2(x)} 1_{N_\sigma^c}(x) dx$. Since $\frac{b}{\sigma^2} 1_{N_\sigma^c} \in L^1_{loc}(\mathbb{R})$, μ is a σ -finite signed measure. By Theorem 4.2, there exists a unique pathwise solution to (11). ■

Example 1 If $N_\sigma \subset N_b$ or N_σ is at most countable, then $\int_0^t 1_{N_\sigma}(X_s) ds = 0$ for all $t > 0$.

4.3 On One SDE with non-sticky boundary conditions

Theorem 4.3 *Consider the following equation*

$$X_t = X_0 + \int_0^t a(X_s) dB_s + \int_0^t b(X_s) ds. \quad (12)$$

Suppose that $a(x)$, $b(x)$ are bounded continuous functions which satisfy the following conditions;

1. *a satisfies (LT) condition,*
2. *$a(0) = 0$, $a(x) \neq 0$ for $x \neq 0$ and $b(0) \neq 0$.*

Then the pathwise uniqueness holds for (12).

In order to lighten the proof, we begin by the following Lemma:

Lemma 4.1 *Under the assumptions of Theorem 4.3, any solution of the equation (12) satisfies*

$$\int_0^t 1_{\{X_s=0\}} ds = 0, \quad \forall t \geq 0, \quad a.s.$$

Proof. Let X be a solution of (12). We have

$$\begin{aligned} L_t^0(X) - L_t^{0-}(X) &= 2 \int_0^t 1_{\{X_s=0\}} dX_s \\ &= 2 \int_0^t 1_{\{X_s=0\}} a(X_s) dB_s + 2 \int_0^t 1_{\{X_s=0\}} b(X_s) ds. \end{aligned}$$

(LT) condition implies that $L_t^0(X) - L_t^{0-}(X) = 0$. Since $a(0) = 0$ and $b(0) \neq 0$, we must have

$$\int_0^t 1_{\{X_s=0\}} ds = 0$$

■

Lemma 4.2 *Consider the following two equations;*

$$X_t^i = X_0^i + \int_0^t a(X_s^i) dB_s + \int_0^t b^i(X_s^i) ds, \quad i = 1, 2, \quad (13)$$

where $a(x)$, $b^i(x)$ are bounded continuous functions.

(i) a satisfies (LT) condition,

(ii) $b^1(x) < b^2(x)$ for any $x \in \mathbb{R}$.

If X^1 and X^2 are solutions of (13) and $X_0^1 \leq X_0^2$, then $X_t^1 \leq X_t^2$ for $\forall t \geq 0$ with probability one.

Proof. Let X^1 and X^2 be solutions of (13) corresponding to b^1 , b^2 respectively.

In case (i) we can Suppose that either b^1 or b^2 are Lipschitz continuous, because it is possible to find a Lipschitz function l such that $b^1 < l < b^2$. By Tanaka's formula:

$$\begin{aligned} (X_t^1 - X_t^2)^+ &= (X_0^1 - X_0^2)^+ + \int_0^t 1_{\{X_s^1 > X_s^2\}} (a(X_s^1) - a(X_s^2)) dB_s \\ &\quad + \int_0^t 1_{\{X_s^1 > X_s^2\}} (b^1(X_s^1) - b^2(X_s^2)) ds + \frac{1}{2} L_t^0(X^1 - X^2). \end{aligned}$$

The function a satisfies (LT) condition, therefore,

$$\psi(t) = \mathbb{E}[(X_t^1 - X_t^2)^+] \leq \mathbb{E} \left[\int_0^t 1_{\{X_s^1 > X_s^2\}} (b^1(X_s^1) - b^1(X_s^2)) ds \right].$$

Thus, if b^1 is Lipschitz with constant C_t ,

$$\psi(t) \leq C_t \cdot \mathbb{E} \left[\int_0^t 1_{\{X_s^1 > X_s^2\}} |X_s^1 - X_s^2| ds \right] = C_t \int_0^t \psi(s) ds,$$

and we conclude by using Gronwall's lemma and the usual continuity arguments.

If b^2 is Lipschitz, using the same arguments, we get:

$$\psi(t) \leq \mathbb{E} \left[\int_0^t 1_{\{X_s^1 > X_s^2\}} |b^2(X_s^1) - b^2(X_s^2)| ds \right].$$

Since $b^1 \leq b^2$, we complete the proof as in the first case. ■

Now, we are ready to give the proof of Theorem.

Proof of Theorem. The proof consists in the construction of the minimal

and the maximal solutions \underline{X} and \overline{X} respectively. This construction is due to Y. Ouknine [21]. Under the assumptions of Theorem 4.3, b is continuous, then there exists a sequence b_n Lipschitz nondecreasing in n , and such that if x_n converges to x , $b_n(x_n)$ converges to $b(x)$. Let X_n be the unique strong solution to the SDE:

$$X_t = X_0 + \int_0^t a(X_s) dB_s + \int_0^t b_n(X_s) ds. \quad (14)$$

By comparison Theorem, the process $X^n = \{X_t^n, t \geq 0\}$ is increasing in n , a.s. For $t \geq 0$, we define $\underline{X}_t = \lim_{n \rightarrow +\infty} X_t^n$.

We denote $X^{\pm K}$ the solutions of the SDE's:

$$X_t = x + \int_0^t \sigma(X_s) dB_s \pm (K+1)t$$

We can assume that $b_n(x) \leq K+1$, $\forall n \in \mathbb{N}$. Thus, $|\underline{X}_t| \leq |X_t^{-K}| + |X_t^{+K}|$, by sample estimations, we deduce that $\mathbb{E} \sup_{s \leq t} |X_s^{\pm K}|^2 < \infty$. Therefore, $|\underline{X}_t| < \infty$ a.s., If we tend $n \rightarrow \infty$, we prove that \underline{X} is a strong solution to the SDE:

$$X_t = X_0 + \int_0^t a(X_s) dB_s + \int_0^t b(X_s) ds. \quad (15)$$

The construction of \overline{X} being treated in similar fashion. It is clear that \underline{X} and \overline{X} have the strong Markov property. So \overline{X} and \underline{X} are diffusion processes with the same local generator

$$L = a^2(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx}$$

and have no stay at the origin by Lemma 4.1. However, by Feller's general theory of one dimensional diffusion processes, there exists only one diffusion process which possesses L as its local generator and does not stay at the origin. Hence the pathwise uniqueness. ■

5 A generalization of the perturbed Tanaka equation

Prokaj in [26] has showed a recent result on the pathwise uniqueness of the so-called perturbed Tanaka equation:

$$Y_t = y + \int_0^t \text{sign}(Y_s) dM_t + N_t \quad (16)$$

Theorem 5.1 (Prokaj 2010) *Suppose that M, N are continuous local martingales with $M_0 = N_0 = 0$ and quadratic and cross-variations that satisfy the condition of orthogonality and domination*

$$\langle M, N \rangle_t = 0 \quad \langle M \rangle_t = \int_0^t q(s) \langle N \rangle_s \quad s \leq t$$

respectively, for some progressively measurable process $q(\cdot)$ with values in a compact interval $[0, c]$. Under these assumptions pathwise uniqueness holds for the perturbed Tanaka equation (16).

In this section, we use the local time technics introduced by Perkins [25] and further developed by LeGall [14], to provide a simple proof of a more general result of this type.

Let $(\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space and $B = (B^{(1)}, B^{(2)})$ be a two dimensional Brownian motion in the filtration $(\mathcal{F}_t)_{t \geq 0}$. We are interested in the uniqueness of the solution for the following equation

$$dX_t = \sigma(X_t) dB_t^{(1)} + \lambda dB_t^{(2)}, \quad (17)$$

where $\lambda \in \mathbb{R}$ is a constant and $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function which satisfies the assumption (2BV) below.

Theorem 5.2 *Suppose that there exists a function f of bounded variation such that for every real numbers x, y*

$$|\sigma(x) - \sigma(y)|^2 \leq |f(x) - f(y)| \quad (2BV)$$

if $\lambda \neq 0$, then the solution of (17) is pathwise unique.

Inspired by Prokaj [26] and [10], we prove a more general statement than Theorem 5.2. Let M, N be two local martingales, we say that M and N are strongly orthogonal martingale if $\langle M, N \rangle = 0$ i.e. whose product is a local martingale.

We say that N dominates M if for some constant $c > 0$ we have $\langle M \rangle \leq c \langle N \rangle$. In other words there is a process Q (it can be chosen to be predictable) such that $\langle M \rangle_t = \int_0^t Q_s d\langle N \rangle_s$ for all $t \geq 0$ and $\mathbb{P}(\forall s \geq 0, 0 \leq Q_s \leq c) = 1$. A localized version of this notion, namely N locally dominates M , holds if this Q is locally bounded.

Theorem 5.3 *Let M, N be continuous local martingales in $(\mathcal{F}_t)_{t \geq 0}$. Assume that M and N are strongly orthogonal and N dominates M . Suppose that there exists a function f of bounded variation such that for every real numbers x, y*

$$|\sigma(x) - \sigma(y)|^2 \leq |f(x) - f(y)|, \quad (2BV)$$

then the solution of the equation

$$dX_t = \sigma(X_t) dM_t + dN_t \quad (18)$$

is pathwise unique.

Proof. Without loss of generality, we shall prove the statement for an increasing function f . Let us first show that $L^0(X - Y) \equiv 0$, whenever X and Y denote any two solutions of the SDE (18) with the same underlying local martingales M and N . By the right continuity of L^0 it is enough to prove that, for any $t \geq 0$,

$$\int_{0+}^{+\infty} \frac{L_t^a(X - Y)}{a} da < +\infty.$$

Indeed, using the density occupation formula we can write for any $\delta > 0$,

$$\int_{0+}^{+\infty} \frac{L_t^a(X - Y)}{a} da = \int_0^t \frac{d\langle X - Y \rangle_s}{X_s - Y_s} 1_{\{X_s - Y_s > 0\}} = \int_0^t \frac{(\sigma(X_s) - \sigma(Y_s))^2}{X_s - Y_s} 1_{\{X_s - Y_s > 0\}} d\langle M \rangle_s.$$

Applying the assumption (BV2) we obtain

$$\int_0^t \frac{(\sigma(X_s) - \sigma(Y_s))^2}{X_s - Y_s} 1_{\{X_s - Y_s > 0\}} d\langle M \rangle_s \leq \int_0^t \frac{|f(X_s) - f(Y_s)|}{X_s - Y_s} 1_{\{X_s - Y_s > 0\}} d\langle M \rangle_s.$$

As a consequence,

$$\mathbb{E} \left[\int_{0+}^{+\infty} \frac{L_t^a(X - Y)}{a} da \right] \leq \mathbb{E} \left[\int_0^t \frac{|f(X_s) - f(Y_s)|}{X_s - Y_s} 1_{\{X_s - Y_s > 0\}} d\langle M \rangle_s \right]. \quad (19)$$

Now, by a localization argument $\|f\|_\infty := \sup_x |f(x)| < \infty$.

Let θ_n denote the standard positive regularizing mollifiers sequence, and define

$$f_n(x) = (\tilde{f}(\cdot) * \theta_n)(x) \quad \text{for } x \in \mathbb{R}, \quad n \in \mathbb{N}^*,$$

where \tilde{f} is any real function such that $\tilde{f}(x) = f(x)$ if $|x| \leq M$ and 0 if $|x| \geq M + 1$.

Note that f_n are increasing functions, with support contained in $[-M - 1, M + 1]$ such that

$$\sup_m \sup_x |f_n(x)| \leq \|f\|_\infty \quad \text{and}; \quad f_n(x) \rightarrow f(x) \quad \text{for every } x \in D^c, \quad |x| \leq M$$

where D is the denumerable set of discontinuous points of the function f . If we denote $Z_t^\alpha = \alpha X_t + (1 - \alpha)Y_t$, the orthogonality of M and N implies that $\langle N \rangle(\cdot) \leq \langle Z^\alpha \rangle(\cdot)$. Let us recall the domination assumption which gives

$$\langle M \rangle(\cdot) \leq c \langle N \rangle(\cdot) \leq c \langle Z^\alpha \rangle(\cdot). \quad (20)$$

Hence, using successively Fatou's Lemma, the intermediate value theorem, and the domination assumption (20) we get,

$$\begin{aligned} & \mathbb{E} \left[\int_0^t \frac{(f(X_s) - f(Y_s))}{X_s - Y_s} 1_{\{X_s - Y_s > 0\}} d\langle M \rangle_s \right] \\ & \leq \liminf_{n \rightarrow +\infty} \mathbb{E} \left[\int_0^t \frac{(f_n(X_s) - f_n(Y_s))}{X_s - Y_s} 1_{\{X_s - Y_s > 0\}} d\langle M \rangle_s \right] \\ & = \liminf_{n \rightarrow +\infty} \mathbb{E} \left[\int_0^t \int_0^1 \frac{\partial f_n}{\partial a}(\alpha X_s + (1 - \alpha)Y_s) d\alpha d\langle M \rangle_s \right] \\ & = \liminf_{n \rightarrow +\infty} \int_0^1 d\alpha \mathbb{E} \left[\int_0^t \frac{\partial f_n}{\partial a}(Z_s^\alpha) d\langle M \rangle_s \right] \\ & \leq c \liminf_{n \rightarrow +\infty} \int_0^1 d\alpha \mathbb{E} \left[\int_0^t \frac{\partial f_n}{\partial a}(Z_s^\alpha) d\langle Z^\alpha \rangle_s \right] \end{aligned}$$

Note that we have used in the first inequality the fact that

$$\int_0^t P[(X_s \in D) \cup (Y_s \in D)] d\langle M \rangle_s = 0. \quad (21)$$

To see that this statement holds, it suffices to remark that

$$\int_0^t 1_{\{X_s=a\}} d\langle M \rangle_s \leq c \int_0^t 1_{\{X_s=a\}} d\langle N \rangle_s \leq \int_0^t 1_{\{X_s=a\}} d\langle X \rangle_s = 0 \quad \forall a \in \mathbb{R}$$

The first inequality due to the domination. The last equality is a consequence of occupation times formula. Hence,

$$\mathbb{E} \left[\int_0^{+\infty} \frac{L_t^a(X-Y)}{a} da \right] \leq \liminf_{n \rightarrow +\infty} \mathbb{E} \left[\int_{\mathbb{R}} \int_0^1 \frac{\partial f_n}{\partial a}(a) L_t^a(Z^\alpha) d\alpha da \right] \quad (22)$$

However, since $\alpha \in (0, 1)$, then standard calculations imply that, for any $p \geq 0$ and $t \geq 0$,

$$\mathbb{E}[\sup_{s \leq t} |X_s|^p] < \infty. \quad (23)$$

Using Tanaka formula and the inequality $|Z_t^\alpha - a| - |Z_0^\alpha - a| \leq |Z_t^\alpha - Z_0^\alpha|$ we deduce that

$$\sup_{\alpha \in [0,1], a \in R} \mathbb{E}[L_t^a(Z^\alpha)] < \infty.$$

Therefore, we obtain

$$\begin{aligned} \mathbb{E} \left[\int_0^{+\infty} \frac{L_t^a(X-Y)}{a} da \right] &\leq c \sup_{\alpha \in [0,1], a \in R} \mathbb{E}[L_t^a(Z^\alpha)] \int_R \frac{\partial f_n}{\partial a}(t, a) da \\ &\leq C \|f\|_\infty \end{aligned}$$

where $C > 0$ is a generic constant. hence $L^0(X-Y) \equiv 0$, by Tanaka formula, we obtain that $|X - Y|$ is a local martingale, thus also a nonnegative supermartingale, with $|X_0 - Y_0| = 0$, and consequently, X and Y are indistinguishable. ■

Remark 5.1 In [10], the authors have showed theorem but under the following elementary comparison:

$$|\sigma(x) - \sigma(y)|^2 \leq \|\sigma\|_{TV} |\sigma(x) - \sigma(y)|$$

where $\|\sigma\|_{TV}$ is the total variation of σ . As we have seen previously, this comparison can be substituted by a weaker assumption (2 BV).

Remark 5.2 We think that the power 2 in the assumption (2 BV) is sharp as in [3].

The equation (17) with a suitably correlated Brownian motion with high variance can restore pathwise uniqueness. The following result can be seen also as a generalization of this equation

$$dX_t = \lambda dt + 1_{\{X_t > 0\}} dW_t + (\eta/2) dV_t, \quad 0 \leq t < \infty$$

considered in [12].

Theorem 5.4 Let A be a process of bounded variation, and W, V are standard Brownian motions, the following equation

$$dX_t = \sigma(X_s) dW_s + (\eta/2) dV_t + dA_t \quad (24)$$

has a pathwise unique solution, provided either

- (i) $\eta \neq [-1, 1]$ and $\langle W, V \rangle_t = (-t/\eta)$, $0 \leq t < \infty$, or
- (i) $\eta \neq 0$ and W and V are independent.

Proof. The equation (24) is equivalent to the following:

$$dX_t = (2\sigma(X_s) - 1) dM_s + N_t + A_t$$

where the process $M := W/2$, $N := (W + \eta V)/2$ are continuous, orthogonal martingales with quadratic variation $\langle M \rangle_t = t/4$ and $\langle N \rangle_t = (\eta^2 - 1)t/4$, respectively.

■

5.1 Some open problems

Let us mention an open problems which we think are quite interesting.

Problem1:

We consider the following stochastic equation

$$X_t = X_0 + B_t + \int_{\mathbb{R}} L_t^a(X) \nu(da) \quad (25)$$

where $L_t^a(X)$ denotes the local time at a for the time t for the semimartingale X , ν is bounded measure on \mathbb{R} . Following Le Gall [15], the solution of (25) is obtained as the limit of a sequence of solutions satisfying:

$$X_t^n = X_0 + B_t + \int_0^t b_n(X_s^n) ds$$

when the measure $\mu^n(da) = b_n(a) da$ converges weakly to some measure μ . The measure ν in (25) is obtained from the limit f of $f_{\mu_n}(x) = \exp(-2 \int_0^x b_n(a) da)$ by this formula:

$$\nu(da) = \frac{f'(da)}{f(a) + f(a^-)},$$

with $f(a^-)$ denotes the left limit of f at a point a . What one can say for the non homogenous case?

$$X_t^n = X_0 + B_t + \int_0^t b_n(s, X_s^n) ds$$

which is equivalent to the sequence of this SDE:

$$X_t^n = X_0 + B_t + \int_0^t \int_{\mathbb{R}} b_n(s, a) d_s L_s^a(X^n) da.$$

Formally, this sequence will converge to :

$$X_t = X_0 + B_t + \int_0^t d_s L_s^{\nu_s}(X) \quad (26)$$

where $L_t^{\nu_s} = \int_{\mathbb{R}} L_s^a(X) \nu_s(da)$.

The pathwise uniqueness of the solution to equation (26) was studied by S. Weinryb [31] in the case $\nu_s(da) = \alpha(s)\delta_0(da)$ where $\alpha : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a deterministic function and $\alpha \leq 1/2$.

In the other hand, we have shown the pathwise uniqueness for (26) when the function α is constant. However, we don't know whether the pathwise uniqueness hold in the other cases.

Problem2: In [6], the authors has studied the weak limit of the following equation:

$$dX_t^\epsilon = b(X_s^\epsilon)ds + \epsilon dB_t,$$

one can ask, what can be the limit of the following equation :

$$dX_t^\epsilon = \sigma(X_s^\epsilon)dM_t + \epsilon dN_t$$

where M and N are two local martingales which satisfy the same assumptions as in section 5.

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